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A NUMERICAL SOLUTION OF THE TWO DIMENSIONAL STEADY-STATE TURBULENT TRANSFER EQUATION

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ABSTRACT

In the finite-difference formulation of the two dimensional steady-state turbulent diffusion equation for solving evaporation problems, two difficulties arise caused by the automatic satisfaction of one of the boundary conditions at the surface and by the infinite size of the solution domain. A general numerical scheme is developed to overcome these difficulties by the use of appropriate transformations. The results of some numerical experiments show that the the longitudinal diffusion term is usually negligible and that, with suitable parameters for roughness and stability, power laws can be as useful for practical solutions as the more complicated logarithmic law.

1. INTRODUCTION

Evaporation as a phenomenon of steady turbulent diffusion of water vapor in air flowing over a free water surface can be described (e.g., Sutton 1934) by

$$u\frac{\partial q}{\partial x}=\frac{\partial}{\partial x}\left(K_{xx}\frac{\partial q}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_{yy}\frac{\partial q}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_{zz}\frac{\partial q}{\partial z}\right) \tag{1}$$

under appropriate boundary conditions. In this equation, u is the mean velocity of the wind in the x direction, q is the specific humidity of the air, and K_{xx}, K_{yy}, and K_{zz} are the longitudinal, lateral, and vertical components of the eddy diffusivity. Due mainly to the mathematical complexity of the available functions describing the wind speed and the diffusivities in which the exact physical nature is far from understood, the complete exact solution of eq (1) for evaporation still seems an almost hopeless task. Possible ways of obtaining approximate solutions are either to aim at analytical solutions after omitting terms from eq (1) that appear negligible and after making simplifying assumptions concerning the wind speed and the diffusivity or alternatively to replace eq (1) by its finite-difference form and to solve the resulting algebraic equations. The former type of solutions has the advantage of providing perhaps more direct insight into the nature of the problem and to be more practical to apply, whereas the latter allows the study of rather complicated physical situations and also the assessment of the errors introduced by various simplifications. Common simplifying assumptions that have been employed in the past to solve evaporation problems are that (1) the wind profile and the diffusivities can be represented by power functions of elevation z, (2) the longitudinal diffusion term is negligible, and (3) the lateral diffusion term is negligible.

It has been shown recently (Brutsaert and Yeh 1969) that, for evaporation from extremely small water surfaces under neutral conditions, lateral diffusion probably contributes less than 13 percent of the average evaporation. Hence, for evaporating surfaces that are several orders of magnitude larger than, say, 1 cm², assumption (3) appears indeed reasonable. However, as far as the authors know, to date no analysis has been performed on the sensitivity of solutions of eq (1) to the use of assumptions (1) and (2).

The main purpose of this paper is to present a numerical scheme of solution of

$$u\frac{\partial q}{\partial x}=\frac{\partial}{\partial x}\left(K_{xx}\frac{\partial q}{\partial x}\right)+\frac{\partial}{\partial z}\left(K_{zz}\frac{\partial q}{\partial z}\right). \tag{1*}$$

The method of solution should be sufficiently general to allow the use of any mathematical form for the wind profile or for the diffusivity as functions of the turbulence structure in the lower atmosphere. As a first result, numerical experiments will then permit the evaluation of the validity of the assumptions mentioned above.

2. PHYSICAL MODEL

The following situation is considered: (1) at the water surface the specific humidity $q=q_w$ is saturated and constant; (2) far away from the water surface (i.e., as $x\rightarrow\pm\infty$), the specific humidity profile $q_a(z)$ is known; (3) at the ground surface of the land adjoining the water body, the vapor flux W_{za} is constant and known. This gives the following boundary conditions:

$$q=q_w \quad \text{at } z=0,\ -x_0<x<x_0, \tag{2}$$

$$q=q_a^\infty \quad \text{at } z\rightarrow\infty,\ -\infty<x<\infty, \tag{3}$$

$$q=q_a(z) \text{ at } z>0,\ |x|\rightarrow\infty, \tag{4}$$

and

$$\rho_a K_{zza}\frac{\partial q}{\partial z}=\rho_a K_{zza}\frac{\partial q_a}{\partial z}=-W_{za} \text{ at } z=0,\quad |x|>x_0, \tag{5}$$

in which ρ_a is the density of the air; K_{zza} is the vertical diffusivity in the air if the water surface were not present; $2x_0$ is the size of the water surface; q_a^∞ is the value of q_a at $z=\infty$; q_a, a function of z only, satisfies

$$0=\frac{\partial}{\partial z}\left(K_{zza}\frac{\partial q_a}{\partial z}\right)=\frac{1}{r}\frac{\partial}{\partial z}\left(K_{zz}\frac{\partial q_a}{\partial z}\right) \tag{6}$$

where $r=K_{zz}/K_{zza}$ will be assumed a constant.

When q is solved from eq (1*), the water vapor flux at the water surface is given by

$$W_z=-\rho_a K_{zz}\frac{\partial q}{\partial z}\bigg|_{z=0}; \tag{7}$$

and the average evaporation is given by

$$E=\int_{-x_0}^{x_0} W_z\, dx/(2x_0). \tag{8}$$

The wind speed is assumed to be a function of z only. Two different laws for the wind profile are of concern here (viz, the power law and the logarithmic law):

$$u=az^m=a_0u_*z^m \tag{9a}$$

and

$$u=\frac{u_*}{\kappa}\ln[(z+z_0)/z_0] \tag{9b}$$

where u_* is the friction velocity, κ is von Kármán's constant, and z_0 is the roughness of the surface. The former was used to study the effect of forward diffusion, and the latter was used to investigate the validity of the assumption of the power law. With u given, K_{zz} can be derived by Reynolds analogy to give from eq (9a)

$$K_{zz}=u_*z^{1-m}/(ma_0) \tag{10a}$$

or from eq (9b)

$$K_{zz}=\kappa u_*(z+z_0). \tag{10b}$$

There is no similar analogy theory to assure a mathematical derivation of K_{xx} since the momentum transfer in the present case is one-dimensional. Nevertheless, it will be assumed (Brutsaert 1967) that

$$K_{xx}=dK_{zz} \tag{11}$$

where d denotes the anisotropy between K_{xx} and K_{zz}.

3. SOLUTION WITH THE POWER LAW

In the straightforward finite-difference representation of eq (1*) with eq (9a) and (10a) subject to boundary conditions (2) through (5), two difficulties arise immediately. First, boundary condition (5) is satisfied automatically when it is evaluated numerically at $z=0$. Second, the numerical domain has to cover the whole upper half plane since boundary conditions (3) and (4) are at infinity.

To overcome the first difficulty, one conveniently introduces the following dimensionless variables and transformations:

$$\xi=x/x_0 \quad \chi=(q-q_a)/(q_w-q_a^0)$$

and

$$\zeta=(mx_0)^{-m/(1+2m)}a_0^{2m/(1+2m)}z^m \tag{12}$$

where q_a^0 is the value of q_a at $z=0$. The purpose of the power m for z is to reduce $z^{1-m}\partial\chi/\partial z$ to $\partial\chi/\partial\zeta$, so that eq (5) is no longer automatically satisfied. To overcome the second difficulty, one transforms the upper half $\xi-\zeta$ plane into a strip in the $\phi-\psi$ plane by means of the following transformation as suggested by Davies (1947) for a different problem:

$$\xi=\cosh\phi\cos\psi$$

and

$$\zeta=\sinh\phi\sin\psi. \tag{13}$$

It can be seen that this transformation brings the upper half plane (i.e., the physical plane of the present problem reaching theoretically all the way to infinity) into a strip of height π, so that the working domain is easier to treat numerically. Thus, a square grid in the strip represents an outwardly increasing grid size in the physical plane; and most of the detail is obtained near the evaporating surface where it is of interest.

With transformations (12) and (13), eq (1*) through (8) become

$$L(\chi)=A\frac{\partial^2\chi}{\partial\phi^2}+2B\frac{\partial^2\chi}{\partial\phi\partial\psi}+C\frac{\partial^2\chi}{\partial\psi^2}+D\frac{\partial\chi}{\partial\phi}+F\frac{\partial\chi}{\partial\psi}=0 \tag{14}$$

where

$$A=\epsilon(\sinh\phi\sin\psi)^{2(1-m)/m}\sinh^2\phi\cos^2\psi+\cosh^2\phi\sin^2\psi,$$

$$B=\sinh\phi\cosh\phi\sin\psi\cos\psi\{1-\epsilon(\sinh\phi\sin\psi)^{2(1-m)/m}\},$$

$$C=\epsilon(\sinh\phi\sin\psi)^{2(1-m)/m}\cosh^2\phi\sin^2\psi+\sinh^2\phi\cos^2\psi,$$

$$D=\frac{\sinh\phi\cosh\phi(2\cos^2\psi\sin^2\psi+\cosh^2\phi\sin^2\psi-\sinh^2\phi\cos^2\psi)}{\sin^2\phi+\sin^2\psi}\times\{\epsilon(\sinh\phi\sin\psi)^{2(1-m)/m}-1\}-(\sinh\phi\sin\psi)^{1/m}\sinh\phi\cos\psi(\sinh^2\phi+\sin^2\psi),$$

$$F=\frac{\sin\psi\cos\psi(2\sinh^2\phi\cosh^2\phi+\sinh^2\phi\cos^2\psi-\cosh^2\phi\sin^2\psi)}{\sinh^2\phi+\sin^2\psi}\times\{\epsilon(\sinh\phi\sin\psi)^{2(1-m)/m}-1\}+(\sinh\phi\sin\psi)^{1/m}\cosh\phi\sin\psi(\sinh^2\phi+\sin^2\psi),$$

$$\epsilon=(m^mx_0^ma_0)^{-4/(1+2m)}d,$$

$$W_z=rW_{za}-\frac{\rho_au_*(q_w-q_a^0)}{(mx_0)^{m/(1+2m)}a_0^{1/(1+2m)}}\frac{\partial\chi}{\partial\phi}\bigg|_{\phi=0}, \tag{15}$$

$$E=rE_a-\frac{1}{2}\frac{\rho_au_*(q_w-q_a^0)}{(mx_0)^{m/(1+2m)}a_0^{1/(1+2m)}}\int_0^\pi\frac{\partial\chi}{\partial\phi}\bigg|_{\phi=0}d\psi, \tag{16}$$

$$\chi=1 \qquad \text{at } \phi=0, \tag{17}$$

$$\chi=0 \qquad \text{at } \phi=\infty, \tag{18}$$

and

$$\frac{\partial\chi}{\partial\psi}=0 \qquad \text{at } \psi=0 \qquad \text{or } \psi=\pi. \tag{19}$$

The derivatives in eq (14) can be approximated by ratios of differences, and these differences are expressible in terms of functional values. For a square net with spacing h, it can be shown (Forsythe and Wasow 1960) that the differential operator L can be approximated by the finite-difference operator L_h to the order of h^2:

$$L_h(\chi)=\frac{1}{h^2}\left\{\left(A+\frac{hD}{2}\right)\chi(\mathrm{E})+\left(A-\frac{hD}{2}\right)\chi(\mathrm{W})\right.$$
$$+\left(C-\frac{hF}{2}\right)\chi(\mathrm{S})+\left(C+\frac{hF}{2}\right)\chi(\mathrm{N})+\frac{B}{2}[\chi(\mathrm{SW})-\chi(\mathrm{SE})$$
$$\left.+\chi(\mathrm{NE})-\chi(\mathrm{NW})]-(2A+2C)\chi(\mathrm{P})\right\}=0 \tag{20}$$

where P denotes the coordinates of the grid point under consideration and N, E, S, W, NE, SE, SW, and NW denote north, east, etc., respectively. Since as a result of transformation (13) the semi-infinite strip has a width π that is an irrational number, it is difficult to choose a grid spacing h to make the whole domain a square net. Therefore, as shown in figure 1, a square net is used throughout the domain except near the upper boundary, $\psi=\pi$, where a partly square net is used. For such a grid (viz, for which $\mathrm{NW}-\mathrm{N}=\mathrm{N}-\mathrm{NE}=\mathrm{W}-\mathrm{P}=\mathrm{P}-\mathrm{E}=\mathrm{SW}-\mathrm{S}=\mathrm{S}-\mathrm{SE}=\mathrm{P}-\mathrm{S}=\mathrm{E}-\mathrm{SE}=h$ and $\mathrm{NW}-\mathrm{W}=\mathrm{N}-\mathrm{P}=\mathrm{NE}-\mathrm{E}=h'$), the following approximation (Yeh 1969) may be used:

$$L_{hh'}(\chi)=\frac{1}{h^2}\left\{\left(A+\frac{hD}{2}\right)\chi(\mathrm{E})+\left(A-\frac{hD}{2}\right)\chi(\mathrm{W})\right.$$
$$+\left[\frac{2Ch^2}{h'(h+h')}+\frac{hh'F}{(h+h')}\right]\chi(\mathrm{N})+\left[\frac{2hC}{(h+h')}-\frac{hh'F}{(h+h')}\right]\chi(\mathrm{S})$$
$$+\frac{hB}{(h+h')}[\chi(\mathrm{SW})-\chi(\mathrm{NW})+\chi(\mathrm{NE})-\chi(\mathrm{SE})]$$
$$\left.-\left[2A+\frac{2hC}{h'}-\frac{(h'-h)hF}{h'}\right]\chi(\mathrm{P})\right\}=0. \tag{21}$$

The boundary condition (19) can be approximated by the finite differences

$$\chi(\mathrm{P})-\chi(\mathrm{S})=0 \quad \text{at} \quad \psi=0 \tag{22}$$

and

$$\chi(\mathrm{P})-\chi(\mathrm{N})=0 \quad \text{at} \quad \psi=\pi. \tag{23}$$

Boundary condition (18) should be taken theoretically at $\phi=\infty$ and requires, therefore, an approximation for the numerical scheme. For the present problem, it was decided to assume that the strip shown in figure 1 extends only up to $\phi=3$. For most problems in the lower atmosphere, this would seem quite satisfactory, since it corresponds in the original physical plane to an elevation or distance about 20 times the size of the evaporating surface.

Apply eq (20) and/or (21) to each interior point of the working domain and eq (17), (18), (22), and (23) to the boundaries of the strip shown in figure 1; a system of linear algebraic equations determining the functional values of the interior points is obtained:

$$A\mathbf{X}=\mathbf{V} \tag{24}$$

where $\mathbf{V}$ is a constant vector in which the components are determined by eq (17) and (18); $\mathbf{X}$ is the unknown vector having the functional values of the interior points

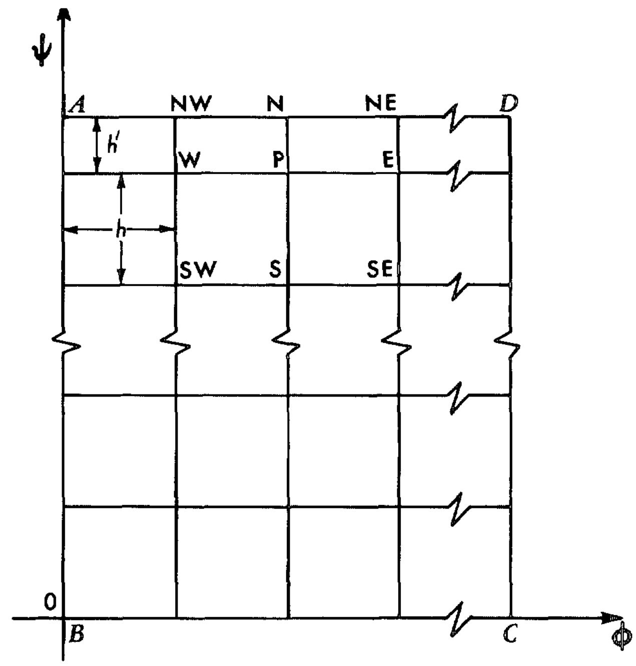

FIGURE 1.—Solution domain for eq (14) represented by a square grid (eq 20) everywhere except near $\psi=\pi$ where a partly square grid (eq 21) is used.

as its components; A is the matrix whose elements are obtained from the finite-difference operators L_h and $L_{hh'}$, and the boundary conditions (22) and (23).

An elimination method was used to solve the system of eq (24). Due to limitations of computer storage, the grid size could not be selected as small as would be desirable to reduce the discretization errors. Therefore, for this purpose, the elimination procedure was applied in conjunction with a successively refined grid size. In other words, initially, a large grid is taken for the whole domain extending to $\phi=3$. This first solution then provides the boundary condition for a much smaller domain extending only between $\phi=0$ and a fraction of $\phi=3$ and with a correspondingly smaller grid size. This new solution obtained with the refined grid is then used again as a boundary condition for the next step, and so on if necessary. Accordingly, the calculations were first carried out with a grid size $h=0.15$ that was eventually refined to 0.075.

The system of eq (24) involves three parameters, m, a_0, and d, which must be known to obtain physically realistic results with the numerical calculations. Very little is known about the parameter d. Yamamoto and Shimanuki (1964) have analyzed experimental data from projects Prairie Grass and Green Glow and found that under neutral conditions the ratio K_{yy}/K_{zz} has a value of about 13. From their study, no simple conclusions can be drawn about the dependence of this ratio on the stability of the air. Nevertheless, the range of K_{yy}/K_{zz} seemed to lie roughly between 1 and 100. Although it is known that K_{zz} is usually somewhat larger than K_{yy}, for the present study in which it is merely intended to evaluate the sensitivity of the solution, this parameter $d=K_{xx}/K_{zz}$ is also assumed to lie in this range. The parameter m is known to have a value of

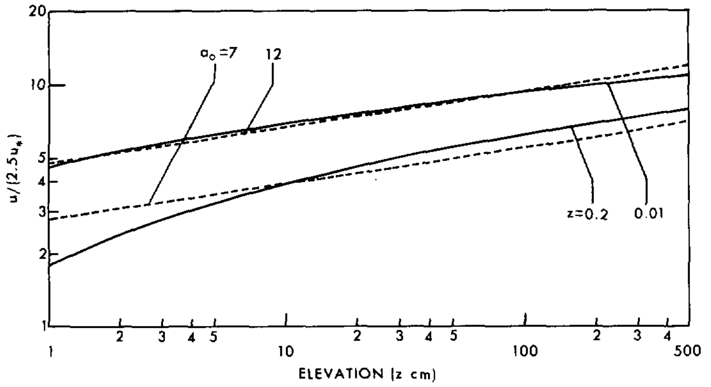

FIGURE 2.—Wind profiles as given by $u/(2.5u_*)$ versus elevation z for the power law and the logarithmic law.

TABLE 1.—*Results of numerical solution of eq (1*) for different m and* x_0 *with* $a_0=9$, $d=10$ *given as* $(E-rE_a)/\{\rho_a u_*(q_w-q_a^0)\}$

m \ $2x_0$	10^{-4}	10^{-2}	10^0	10^2	10^4	10^6
1/6	0.5924	0.2933	0.1686	0.0956	0.0538	0.0303
1/7	.4632	.2572	.1598	.0966	.0580	.0348
1/8	.3851	.2326	.1531	.0976	.0616	.0389

TABLE 2.—*Results of numerical solution of eq (1*) for different* a_0 *and* x_0 *with* $m=1/7$, $d=10$ *given as* $(E-rE_a)/\{\rho_a u_*(q_w-q_a^0)\}$

a_0 \ $2x_0$	10^{-4}	10^{-2}	10^0	10^2	10^4	10^6
7	0.5906	0.3175	0.1913	0.1174	0.0705	0.0423
11	.3829	.2212	.1375	.0827	.0496	.0297

TABLE 3.—*Results of numerical solution of eq (1*) for different d and* x_0 *with* $a_0=9$, $m=1/7$ *given as* $(E-rE_a)/\{\rho_a u_*(q_w-q_a^0)\}$

d \ $2x_0$	10^{-4}	10^{-2}	10^0	10^2	10^4	10^6
1	0.4292	0.2674	0.1612	0.0967	0.0580	0.0349
100	.5352	.2819	.1545	.0954	.0579	.0348

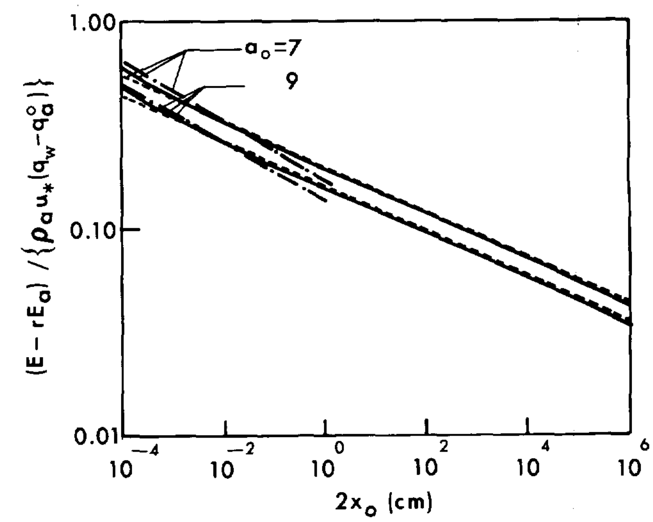

FIGURE 3.—Reduced evaporation rate versus size of the evaporating surface as obtained by numerical solution of eq (1*, solid line), eq (25, dashed line), and eq (26, dash-dot line) for $m=1/7$, $a_0=7$ and 9, and $d=10$.

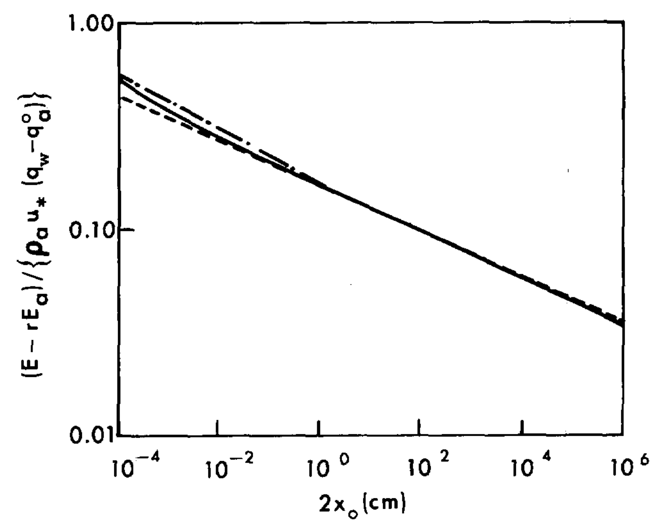

FIGURE 4.—Same as figure 3, but for $m=1/7$, $a_0=9$, and $d=100$.

1/7 under neutral conditions and to decrease with decreasing stability. Although this is an admittedly rough approximation, small deviations from neutral stability can thus be represented by values of m that deviate slightly from 1/7. The parameter a_0 depends mostly on the roughness of the surface and also other factors such as the stability of the air. For neutral conditions, it can be obtained by matching the power profile with the logarithmic profile. Of course, as illustrated in figure 2, the problem is where to match the two curves. This matter will be further discussed below.

The calculations were carried out for combinations of three different values of each parameter m, a_0, and d. The results are given in tables 1, 2, and 3 and in figures 3 and 4. To determine the effect of the longitudinal diffusion term, one must compare the solution of eq (24) with the solution of the equation in which this term is neglected. Fortunately, however, for this case [viz, for eq (1*) but without the first term on the right-hand side], an analytical solution is available, which was first obtained by Sutton (1934) using a different physical model for u and K_{zz}. Adaptation of that solution to the present physical model and transformation into a simpler form (cf. Frost 1946, p. 24) gives

$$E-rE_a=\rho_a u_*(q_w-q_a^0)/\{\Gamma(\nu)\nu^{1-2\nu}(1-\nu)m^\nu x_0^\nu a_0^{1-2\nu}\} \quad (25)$$

where $\nu=m/(1+2m)$. The results of calculations performed for some of the same values of the parameters used in the solution of eq (1*) are shown in figures 3 and 4. Comparison of these two sets of results shows the rather puzzling fact that the inclusion of the longitudinal diffusion term seems to decrease the evaporation. The negative effect of longitudinal diffusion is also confirmed by a perturbation solution of eq (1*) that will appear in a later communication. This puzzling result may perhaps be explained by the fact that, while forward diffusion provides one mechanism for mass transfer, it may also damp the mass transfer by advection. At any rate, the net effect of forward diffusion is very small; and as seen from this numerical solution (figs. 3 and 4), it can be neglected.

Actually, for large evaporating surfaces, this result is to be expected from dimensional reasoning. However, dimensional reasoning only gives a qualitative idea and does not permit the exact determination of the lower limit on the size of a "large" area without the experiments or calculations of the present type. Similarly, for extremely small surfaces, the same dimensional reasoning (Brutsaert 1967) leads to the conclusion that, in eq (1*), the advection term is negligible as compared to the longitudinal eddy diffusion term. The solution of this case under the present boundary conditions and for the present physical model (Brutsaert and Yeh 1969) gives

$$E-rE_a=\rho_a u_*(q_w-q_a^0)2\pi d^{m/2}\Gamma(1-m/2) \div \{m(1-m)\Gamma^2[(1-m)/2]\Gamma(m/2)a_0\}. \tag{26}$$

To determine what exactly should be meant by an "extremely small" surface, the results obtained with this equation are compared in figures 3 and 4 with the numerical results obtained from the complete eq (1*). This shows that the two solutions tend to come together for decreasing x_0. However, as such, the extremely small surface is probably never encountered in nature. Nevertheless, this case is of theoretical interest since it shows the general tendency of the parameters for decreasing evaporating surface areas; and it allowed the evaluation of the effect of lateral eddy diffusion under extreme conditions.

To check the validity of the comparisons shown in figures 3 and 4, it was necessary to determine the accuracy of the numerical scheme for solving eq (1*). This was done by omitting the longitudinal diffusion term [i.e., by putting ϵ in eq (14) equal to zero]. The results of calculations are listed in table 4, and comparison with the analytical solution (eq 25) shows that the total error in the numerical scheme is probably smaller than 1 percent.

4. SOLUTION WITH THE LOGARITHMIC LAW

After having shown that forward diffusion is negligible, it is the purpose of this section to test the validity of the power law for the wind profile in the prediction of evaporation from large water surfaces. Equation (1*), neglecting the longitudinal term, must be solved with $u(z)$ and K_{zz} given by eq (9b) and (10b); the results must be compared with the analytical solution (25) or with the numerical solution for $m=1/7$ with $\epsilon=0$.

For comparison with the earlier numerical results obtained through transformation (12) with $m=1/7$, the following transformation is introduced:

$$\xi=x/x_0;\qquad \chi=(q-q_a)/(q_w-q_a^0);$$
$$\zeta=(z_0/x_0\kappa^2)^{1/9}\ln[(z+z_0)/z_0]. \tag{27}$$

From here on, following the same procedures and transformations as those used in the numerical solution with the power law for $\epsilon=0$, we obtained the numerical results given in table 5. In figure 5, these data are compared with those obtained by means of eq (25) based on the power law. The results show that, for most values of x_0 of interest, the logarithmic law gives evaporation values that are approximately equal to those given by the power law, provided that z_0 and a_0 are chosen such that the two wind profiles are matched as shown in figure 2.

The relationship between a_0 and z_0 is now analyzed. Table 6 shows the variation of evaporation with a_0 for $m=1/7$. Table 7 gives the dependence of evaporation on z_0 for a certain lake size, $2x_0=10^4$ cm. The former is obtainable with the power law for the wind profile either by the numerical method for $\epsilon=0$ or by the analytical solution (eq 25). The latter is obtained with the logarithmic law by the numerical method. From these results that are plotted in figure 6, figure 7 is plotted showing a_0 versus z_0 for constant values of evaporation. This shows that, under neutral conditions, the power law for the wind profile is valid in the prediction of evaporation from lakes provided that a_0 is chosen from figure 7 for a given z_0. Looking back to figure 2, one sees that, with a pair of a_0 and z_0 from figure 7, the two profiles match between 1 cm and 5 m. Furthermore, it is seen that, for larger z_0, the intersection point of the two profiles is at a higher elevation. From figure 7, an empirical formula

TABLE 4.—*Comparison between numerical solution of eq (1*) without the x diffusion term and the analytical solution (25) with* $m=1/7$, $a_0=9$, $d=10$ *given as* $(E-rE_a)/\{\rho_a u_*(q_w-q_a^0)\}$

$2x_0$	10^{-4}	10^{-2}	10^{0}	10^{2}	10^{4}	10^{6}
Numerical solution	0.4537	0.2720	0.1631	0.0978	0.0586	0.0351
Analytical solution	.4560	.2733	.1639	.0982	.0589	.0353
Error in percent	−.505	−.478	−.336	−.407	−.510	−.566

TABLE 5.—*Results of numerical solution of eq (1*) neglecting x diffusion with the logarithmic law for different values of* z_0 *and* x_0 *given as* $(E-rE_a)/\{\rho_a u_*(q_w-q_a^0)\}$

z_0 \ $2x_0$	10^{-2}	10^{0}	10^{2}	10^{4}	10^{6}	10^{8}
0.01	0.3212	0.1476	0.0863	0.0518	0.0309	0.0192
20	.6415	.2341	.1211	.0722	.0433	.0252

can be derived relating a_0 with z_0 in the range of interest, namely

$$a_0=5.5\,z_0^{-1/7}, \tag{28}$$

suggested by the fact that a_0 has dimension $(L^{-1/7})$. Equation (28) is valid only for neutral conditions. Thus, for the general case, a relationship between a_0 and z_0 might be postulated of the type

$$a_0=Cz_0^{-m} \tag{29}$$

where C amd m would now mainly depend on the stability of the atmosphere. Equations similar to eq (28) have been proposed in the past. Frost (1946) from mixing length considerations reasoned that $C=1/m$, or $C=7$ under neutral conditions. Calder (1949, p. 165) calculated values of C lying between 4 and 7 and values of m lying between 1/4.5 and 1/8 for diffusion from a line source. At any rate, these different values only confirm that the present numerically obtained result (eq 28) should provide a reasonable basis for practical evaporation calculations.

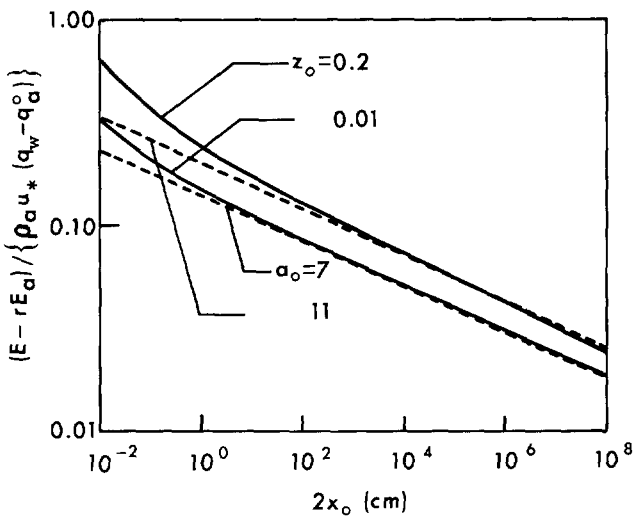

FIGURE 5.—Reduced evaporation rate versus size of the evaporating surface as obtained by numerical solution of eq (1*) neglecting forward diffusion for the logarithmic law (solid line) and by analytical solution (eq 25) for the power law (dotted line).

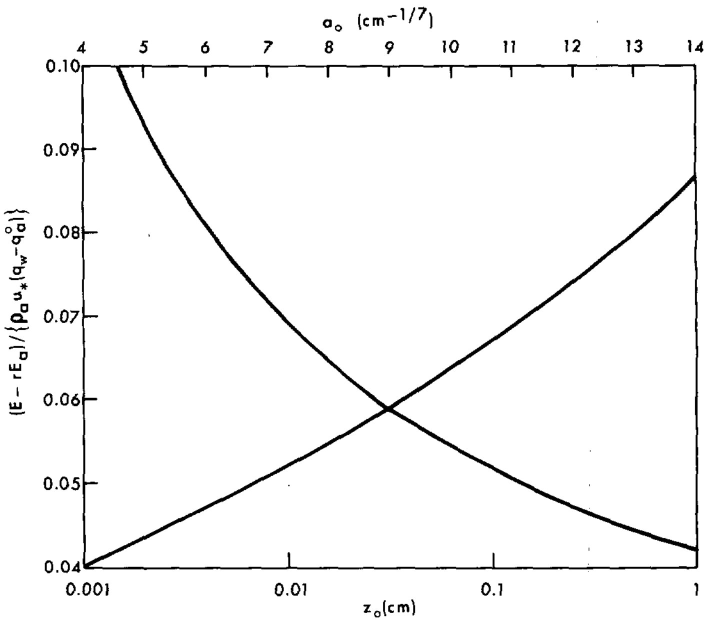

FIGURE 6.—Variation of the reduced evaporation rate with a_0 and with z_0 as obtained by means of analytical solution (eq 25) for the power law and by means of the numerical solution neglecting forward diffusion for the logarithmic law, respectively, for $m=1/7$ and $2x_0=10^4$.

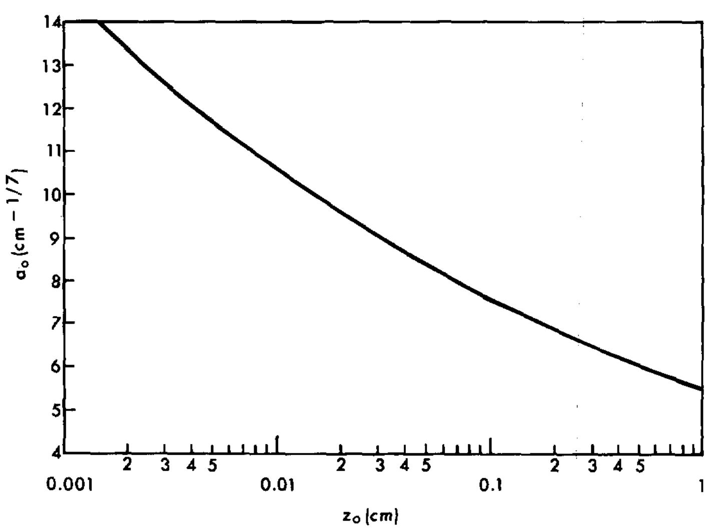

FIGURE 7.—Relationship between$_0$ a and z_0 obtained from figure 6.

TABLE 6.—*Variation of evaporation with a_0 determined from the analytical solutions (eq 25); these data are also obtainable from numerical solution of eq (1*) neglecting x diffusion.*

a_0	4	5	6	7	8	9	10	11	12	13
$(E-rE_a)/\{\rho_a u_*(q_w-q_a^0)\}$	0.1106	0.0930	0.0807	0.0716	0.0645	0.0589	0.0543	0.0504	0.0471	0.0442

TABLE 7.—*Variation of evaporation with z_0 determined from the numerical solutions of eq (1*) neglecting x diffusion*

z_0	0.001	0.002	0.004	0.01	0.02	0.04	0.1	0.2	0.4	1.0
$(E-rE_a)/\{\rho_a u_*(q_w-q_a^0)\}$	0.0402	0.0435	0.0470	0.0520	0.0562	0.0606	0.0671	0.0725	0.0783	0.0866

5. CONCLUSIONS

The two successive transformations of the independent variables of eq (12) or (27) and (13) provide a convenient way of avoiding the numerical difficulties inherent in the two-dimensional type of evaporation or cooling boundary conditions of the present problem. Comparison with the exact solution for a simple case shows that the error in the proposed method is probably smaller than 1 percent.

The numerical experiments with this scheme show that longitudinal diffusion is negligible as compared to wind advection if the evaporating surface is larger than, say, 10 cm. They also show that the power law can be as useful as the logarithmic law under near-neutral conditions provided a_0 is chosen according to figure 7 or eq (28).

Further investigations are required on the validity of the power law under diabatic conditions. Nevertheless, it is expected that the power law will give good results in the prediction of evaporation if the power index m and the coefficient a_0 are chosen such that it matches the experimental wind profile or other more sophisticated laws describing the wind profile.

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REFERENCES

Brutsaert, Wilfried, "Evaporation From a Very Small Water Surface at Ground Level: Three-Dimensional Turbulent Diffusion Without Convection," *Journal of Geophysical Research*, Vol. 72, No. 22, Nov. 15, **1967**, pp. 5631–5639.

Brutsaert, Wilfried, and Yeh, Gour-Tsyh, "Evaporation From an Extremely Narrow Wet Strip at Ground Level," *Journal of Geophysical Research, Oceans and Atmospheres*, Vol. 74, No. 13, June 20, **1969**, pp. 3431–3433.

Calder, K. L., "Eddy Diffusion and Evaporation in Flow Over Aerodynamically Smooth and Rough Surfaces: A Treatment Based on Laboratory Laws of Turbulent Flow With Special Reference to Conditions in the Lower Atmosphere," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. 2, Oxford University Press, London, England, **1949**, pp. 153–176.

Davies, D. R., "Turbulence and Diffusion in the Lower Atmosphere With Particular Reference to the Lateral Effect," *Proceedings of the Royal Society of London*, Ser. A, Vol. 190, No. 1021, England, July 8, **1947**, pp. 232–244.

Forsythe, G. E., and Wasow, W. R., *Finite Difference Methods for Partial Differential Equations*, John Wiley & Sons, Inc., New York, N.Y., **1960**, 444 pp.

Frost, R., "Turbulence and Diffusion in the Lower Atmosphere," *Proceedings of the Royal Society of London*, Ser. A, Vol. 186, England, **1946**, pp. 20–35.

Sutton, Oliver Graham, "Wind Structure and Evaporation in a Turbulent Atmosphere," *Proceedings of the Royal Society of London*, Ser. A, Vol. 146, England, **1934**, pp. 701–722.

Yamamoto, Giichi, and Shimanuki, Atsushi, "The Determination of Lateral Diffusivity in Diabatic Conditions Near the Ground From Diffusion Experiments," *Journal of the Atmospheric Sciences*, Vol. 21, No. 2, Mar. **1964**, pp. 187–196.

Yeh, Gour-Tsyh, "Turbulent Diffusion by Forced Convection," Ph. D. thesis, School of Civil Engineering, Cornell University, Ithaca, N.Y., **1969**, 124 pp.

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